

TWO QUICK DERIVATIONS OF THE BLACK-SCHOLES OPTION PRICING FORMULA

Mark Davis

Mathematics Institute, University of Oslo, P.O. Box 1053 Blindern, N-0315 Oslo 3
and

Department of Electrical Engineering, Imperial College, London SW7 2BT, England

1

1. Introduction

The Black-Scholes formula for the price of a European call option may quite possibly be the most widely-applied result in stochastic analysis, since it is in routine day-to-day use in every traded options market. Since its original discovery in 1973, an enormous amount of work has appeared giving explanations, justifications, extensions and alternative proofs of the formula, and much of this concerns matters which are of interest in stochastic analysis *per se*, independent of the capitalist origins of the problem. Of course, this puts the subject firmly in the grand tradition of probability theory as a whole, which is rooted—as Leo Breiman remarks in the Introduction to his book *Probability*—equally in measure theory and in gambling.

There are two basic approaches to the Black-Scholes formula: an *arbitrage* argument along the lines used by Black and Scholes themselves, and a *martingale representation* approach. Both are essential in gaining a full understanding of the formula. This note attempts to give the quickest possible proofs from these two points of view.

2. Options and Stock Price Models

Let $y(t)$ denote the price of one share of a certain stock at time t . A *European call option* on the stock, written at time 0, is the right to buy one share at a specified time T in the future and at a specified price c (the *exercise price*). If in fact the stock price $y(T)$ at time T turns out to be greater than c then one can make a profit of $y(T) - c$ by buying at the exercise price and immediately reselling. On the other hand if $y(T) \leq c$ then the option is worthless and will not be exercised. Thus by holding the option one acquires at time T a payoff of $(y(T) - c)^+$. The option pricing problem is to determine a fair price that one should pay at time zero to acquire the option. (Certainly the price is positive, since there is no “downside risk”). On the face of it, since purchase of an option is a speculative venture, one would think that a fair price would depend on the attitude to risk of the person to whom the option is being sold, but this is not so: Black and Scholes [1] showed that there is a unique “fair price” that will be agreed on by all investors. Our aim is to explain this result.

Of course, to say anything about fair prices for options requires us to have a probabilistic model describing the evolution of the stock price $y(t)$. The model used by Black and Scholes—and by the vast majority of other authors on these topics—is *geometric Brownian motion*, described as follows. Let $(z(t))_{t \geq 0}$ be a standard Brownian motion process defined on some stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. This means that $z(t)$ is a gaussian process with $z(0) = 0$ and with independent increments, an increment $(z(s) - z(t))$ having mean zero and variance $|t - s|$. Then $y(t)$ is the solution of the *stochastic*

differential equation

$$(1) \quad y(t) - y(0) = \int_0^t \alpha y(s) ds + \int_0^t \sigma y(s) dz(s).$$

This is more intuitive if written in “differential” form as

$$(2) \quad dy(t) = \alpha y(t)dt + \sigma y(t)dz(t),$$

or, for a small time increment Δt :

$$\Delta y = y(t)[\alpha \Delta t + \sigma \Delta z],$$

and we recall that $\Delta z \sim N(0, \Delta t)$. If $\sigma=0$, $y(t)$ simply grows at constant interest rate α . When $\sigma > 0$, the interest rate is perturbed by an independent sequence of increments Δz (which may, of course, be negative).

We can get (2) in explicit form by applying the Ito formula to calculate $\log(y(t))$. We find that

$$\begin{aligned} d(\log(y(t))) &= \frac{1}{y} dy - \frac{1}{2} \frac{1}{y^2} \sigma^2 y^2 dt \\ &= \alpha dt + \sigma dz - \frac{1}{2} \sigma^2 dt. \end{aligned}$$

Thus

$$(3) \quad y(t) = y(0) \exp\left\{\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma z(t)\right\}$$

An obvious question is whether this is actually an accurate model for real stock prices. It is surprisingly hard to give a clear-cut answer to this question. The over-riding advantage of the model is that it only has two parameters, α and σ ; as we shall see later, the Black Scholes formula is independent of α , leaving us with only a single parameter—the stock *volatility* σ —to estimate. Even though empirical evidence does not convincingly support (3) as an accurate model, at least over long time periods, any rival model will certainly involve more parameters and *its* justification becomes correspondingly more problematical. A discussion of these issues will be found in [2]; in general, practitioners seem happy with (3) as a reasonable and workable approximation to reality.

3. Option Pricing — the Black-Scholes Formula

Let us consider a financial market in which there is a stock satisfying (3) above and a riskless bond with constant rate of interest r , so that the bond price $x(t)$ satisfies

$$(4) \quad dx(t) = rx(t).$$

Consider a sequence of times $0=t_0 < t_1 < t_2 \dots < t_n=T$, and suppose an investor holds ϕ_k units of stock in the interval $]t_k, t_{k+1}]$, $k=0,1,\dots,n-1$. The capital gain over the interval $]t_k, t_{k+1}]$ is $\phi_k(y(t_{k+1}) - y(t_k))$. If we define a function $\phi(t)$ by

$$(5) \quad \phi(t) = \phi_k, t \in]t_k, t_{k+1}],$$

then the total capital gain over the interval $[0, T]$ is

$$\sum_{k=0}^{n-1} \phi_k (y(t_{k+1}) - y(t_k)) = \int_0^T \phi(t) dy(t).$$

$\phi(t)$ is called a *trading strategy*. We may allow our choice of ϕ_k to depend on the observed prices $\{y(t), 0 \leq t \leq t_k\}$, in which case $\phi(t)$ is a stochastic process which is “nonanticipative”, i.e. independent of future increments of the basic Brownian motion process $z(t)$. We now generalize and define a trading strategy to be *any* nonanticipative process (not necessarily a “step function” as in (5).) In doing this we are implicitly assuming that the market is what is called “frictionless”, which means that trading can take place in any amounts (not necessarily integral numbers of shares) at any time without any transaction costs; also that holdings can be negative and that interest rates are the same on borrowing and lending. Of course, real markets are not frictionless, but making this assumption leads to a viable theory which usefully approximates reality.

Let $\phi(t)$ be as above, and $\psi(t)$ be a trading strategy in the riskless bond. The total value of our portfolio at time t is then

$$(6) \quad V(t) = \phi(t)y(t) + \psi(t)x(t).$$

On the other hand the cumulative capital gains over the interval $[0, T]$ are

$$(7) \quad \int_0^T \phi(t) dy(t) + \int_0^T \psi(t) dx(t)$$

The strategy (ϕ, ψ) is *self-financing* if for all $T > 0$,

$$(8) \quad V(T) - V(0) = \int_0^T \phi(t) dy(t) + \int_0^T \psi(t) dx(t),$$

i.e. if the increase in capital value is entirely due to the accumulated gains in trading and does not involve injection of funds from outside beyond the initial capital $V(0)$ at time zero.

An *arbitrage opportunity* is the existence of a self-financing strategy such that $V(0)=0$ but $\text{Prob}(V(T) > 0) > 0$, i.e. a riskless strategy for capital gain (otherwise known as a “free lunch”). It is axiomatic that, in frictionless markets, there cannot be arbitrage opportunities. This implies in particular that any two riskless investments must have the same rate of interest; otherwise one could create free lunches by borrowing from one to invest in the other.

The basis of option pricing theory is this: Black and Scholes [1] showed that in a frictionless market it is possible to devise a self-financing strategy that completely hedges the risk of the option, i.e. such that the value $V(T)$ of the portfolio at the exercise time T exactly coincides with the value $(y(T) - c)^+$ of the option. The “fair price” for the option is therefore simply the initial capital required to operate the hedging strategy. The argument goes as follows.

Suppose the price of the option at time $t < T$ is some as yet undetermined function $P(t, y(t))$ of the time and the current stock price $y(t)$. Let (ϕ, ψ) be a self-financing strategy, and consider forming a portfolio which consists of the option (on one share) together with the holdings in stock and bond dictated by the strategy (ϕ, ψ) . The value of this portfolio is then

$$W(t) = P(t, y(t)) + V(t)$$

where $V(t)$ is given by (6). Using the self-financing property (8) we see that

$$W(t) = P(t, y(t)) + V(0) + \int_0^t \phi(s) dy(s) + \int_0^t \psi(s) dx(s).$$

Now expand $P(t, y(t))$ by the Ito formula and use the stock and bond equations (2), (4) to get

$$(9) \quad dW(t) = (P_t + P_y \alpha y(t) + \frac{1}{2} P_{yy} \sigma^2 y^2(t)) dt + P_y \sigma y(t) dz(t) \\ + \phi(t) \alpha y(t) dt + \phi(t) \sigma y(t) dz(t) + \psi(t) \sigma x(t) dt.$$

Here $P_y = \partial P / \partial y$ etc and all such terms are evaluated at $(t, y(t))$. Now let us make the following choice for the strategy ϕ :

$$(10) \quad \boxed{\phi(t) = -P_y(t, y(t))}$$

Then all the terms involving P_y in (9) cancel, and we obtain

$$(11) \quad dW(t) = \{P_t + \frac{1}{2} P_{yy} \sigma^2 y^2(t) + \psi(t) r x(t)\} dt.$$

We have thus created a “locally riskless” strategy (no “dz” terms). By the arbitrage argument, this portfolio must have the constant riskless growth rate r . So $W(t)$ must satisfy

$$(12) \quad dW(t) = rW(t) dt = r(P(t, y(t)) + V(t)) dt \\ = r(P(t, y(t)) + \phi(t)y(t) + \psi(t)x(t)) dt.$$

Comparing (11) and (12) and using (10) we find that

$$P_t + \frac{1}{2} P_{yy} \sigma^2 y^2(t) + \psi(t) r x(t) = rP - rP_y y(t) + r\psi(t)x(t).$$

This will be satisfied if $P(t, y)$ satisfies the *partial differential equation* (PDE)

$$(13) \quad \boxed{\frac{\partial}{\partial t} P(t, y) + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2}{\partial y^2} P(t, y) + r y \frac{\partial}{\partial y} P(t, y) - rP(t, y) = 0}$$

We also know that the value of the option at time T is just $(y(T) - c)^+$, so that

$$(14) \quad \boxed{P(T, y) = (y - c)^+}$$

But the PDE (13) with boundary condition (14) has a unique solution; in fact the solution is

$$(15) \quad P(t, y) = y\Phi(g(T-t, y)) - ce^{-r(T-t)}\Phi(h(T-t, y))$$

where $\Phi(x)$ is the standard normal distribution function and

$$g(t,y) = (\ln(\frac{y}{c}) + (r + \frac{1}{2}\sigma^2)t)/\sigma\sqrt{t}$$

$$h(t,y) = g(t,y) - \sigma\sqrt{t}.$$

Let us now consider a trading strategy (φ, ψ) in which

$$(16) \quad \begin{aligned} \varphi(t) &= P_y(t, y(t)) \quad (= -\phi(t)) \\ \psi(t) &= \frac{1}{x(t)}(P(t, y(t)) - y(t)P_y(t, y(t))) \end{aligned}$$

with initial capital $\mathcal{V}(0) = P(0, y(0))$, where P is given by (15). We claim:

- i) The value $\mathcal{V}(t)$ is equal to $P(t, y(t))$ almost surely for all $t \in [0, T]$.
- ii) The strategy (φ, ψ) is self-financing.

This is “perfect hedging”: provided with initial capital $p := P(0, y(0))$, the investor can form a portfolio whose value at time T is guaranteed to be exactly $(y(T) - c)^+$. Thus p is the “fair price” of the option, in that (a) the investor is indifferent between being paid p at time 0 or $(y(T) - c)^+$ at time T , and (b) if the option price were anything other than p then either the buyer or the seller could make an arbitrage (= risk-free) profit.

To show (i), note that it holds by definition at $t = 0$, and that with $\varphi = P_y$ the incremental capital gain is

$$\begin{aligned} d\mathcal{V}(t) &= \varphi dy + \psi dx \\ &= P_y \alpha y(t) dt + P_y \sigma y(t) dz(t) + \psi(t) r x(t) dt \end{aligned}$$

On the other hand, the increment in the option value is

$$dP(t, y(t)) = (P_t + P_y \alpha y(t) + \frac{1}{2} P_{yy} \sigma^2 y^2(t)) dt + P_y \sigma y(t) dz(t).$$

If

$$(17) \quad \psi(t) = \frac{1}{rx(t)}(P_t + \frac{1}{2} P_{yy} \sigma^2 y^2(t)),$$

then these increments coincide and hence $\mathcal{V}(t) = P(t, y(t))$ for all t . Using (13), we see that (16) and (17) are the same. We need to verify that the strategy (φ, ψ) is self-financing; this is the case if

$$d(\varphi(t)y(t) + \psi(t)x(t)) = \varphi(t)dy(t) + \psi(t)dx(t).$$

The right-hand side of this equation is equal to $dP(t)$, from the above, and the left-hand side is equal to

$$d(P_y y(t)) + d(P - P_y y(t)) = dP(t),$$

using (16). This completes the proof.

Remark: As noted earlier, a key feature of the Black-Scholes formula is that it does not depend on α , the mean growth rate of the stock. The only two parameters involved are the riskless interest rate r , which is “known” and the so-called *volatility* σ . From the statistical point of view this is fortunate, because it turns out that σ is much easier to estimate from data than α .

4. A "Martingale" Approach

There is a probabilistic representation for the solution $P(t, y)$ of the Black-Scholes partial differential equation (3.8), based on the "Feynman-Kac" formula. This is

$$(18) \quad P(t, y) = E[e^{-r(T-t)}(\xi(T) - c)^+]$$

where $\xi(t)$ satisfies the stochastic differential equation

$$(19) \quad \begin{aligned} d\xi(s) &= r\xi(s)ds + \sigma\xi(s)dz(s), \\ \xi(t) &= y. \end{aligned}$$

Note that this is similar to the stock price equation (2) except that the growth rate α has been replaced by the riskless rate r . An independent derivation of (18) can be given as follows. To start with, it is convenient now to parametrize strategies in terms of their *monetary value* rather than *number of shares*. Let $w(t)$ be the total value of a portfolio and $\pi(t)$ value of the holdings in stock; thus $w(t) - \pi(t)$ is held in the bond and the incremental growth of wealth is given by

$$\begin{aligned} dw(t) &= \pi(t)(\alpha dt + \sigma dz(t)) + (w(t) - \pi(t))r dt \\ \text{i.e.} \\ (20) \quad dw(t) &= rw(t)dt + (\alpha - r)\pi(t)dt + \pi(t)\sigma dz(t). \end{aligned}$$

A trading strategy is *admissible* only if $w(t) \geq 0$ for all t . Define $\vartheta = (\alpha - r)/\sigma$ and the process $\tilde{z}(t) := z(t) + \vartheta t$. In terms of $\tilde{z}(t)$, (4.2) becomes

$$(21) \quad dw(t) = rw(t)dt + \pi(t)\sigma d\tilde{z}(t),$$

while the stock price process $y(t)$ satisfies

$$dy(t) = ry(t)dt + \sigma y(t)d\tilde{z}(t),$$

to which, as in §2, the explicit solution with initial condition $y(0) = y_0$ is

$$y(t) = y_0 \exp\{(r - \frac{1}{2}\sigma^2)t + \tilde{z}(t)\} =: h_t(\tilde{z}(t)).$$

The inverse relationship is

$$\tilde{z}(t) = h_t^{-1}(y(t)) = \frac{1}{\sigma} \ln \left(\frac{y(t)}{y_0} \right) - \frac{1}{\sigma} (r - \frac{1}{2}\sigma^2)t.$$

The value of the option at the exercise time T is therefore $(y(T) - c)^+ = (h_T(\tilde{z}(T)) - c)^+$. We claim that there is a trading strategy π such that $w(T) = (y(T) - c)^+$ a.s. Indeed, let $v(t, z)$ be the unique solution of the heat equation

$$(22) \quad \begin{aligned} v_t - rv + \frac{1}{2}v_{zz} &= 0, \\ v(T, z) &= (h_T(z) - c)^+, \end{aligned}$$

and define $w(t) = v(t, \tilde{z}(t))$. By the Ito formula, $w(t)$ then satisfies (21) with

$$(23) \quad \pi(t) = \frac{1}{\sigma} v_z(t, \tilde{z}(t)), \quad w(0) = v(0, 0).$$

Thus if we apply trading strategy π with initial capital $w(0)$ as in (23), the value of the option will be

exactly reproduced at time T . Thus $v(0,0)$ must coincide with the Black-Scholes price. To verify this, define

$$(24) \quad \mathcal{P}(t, y) = v(t, h_t^{-1}(y)).$$

Then $\mathcal{P}(T, y) = (y - c)^+$, and an easy calculation shows that \mathcal{P} satisfies the Black-Scholes partial differential equation (13). Hence $\mathcal{P}(t, y) = P(t, y)$ and $v(0, 0) = P(0, h_0(0)) = P(0, y_0)$, which is the Black-Scholes price since we assumed that $y(0) = y_0$.

Finally, we want to justify the representation (18) obtained from the Feynman-Kac formula. This justification depends on the so-called Cameron-Martin-Girsanov formula, which states that if we define a new probability measure on the path space Ξ for $z(t)$ by the formula

$$\tilde{P}(A) = \int_A \exp(-\vartheta z(t) - \frac{1}{2}\vartheta^2(t)) P(dz)$$

then the process $\tilde{z}(t)$ is a Brownian motion under measure \tilde{P} . If we expand $v(t, \tilde{z}(t))$ by the Ito formula, using (22), we obtain

$$\begin{aligned} v(0, 0) &= \tilde{E}[e^{-rT} v(T, \tilde{z}(T))] \\ &= \tilde{E}[e^{-rT} (h_T(\tilde{z}(T)) - c)^+] \\ &= \tilde{E}[e^{-rT} (y(T) - c)^+], \end{aligned}$$

and this is the same as the representation (18) derived from the Feynman-Kac formula, since $y(T)$ has the same distribution under measure \tilde{P} as the distribution of $\xi(T)$ defined by (19). This expresses the option price as the expected discounted value of the option at the exercise time T , but note that it is the expected value under the *risk-neutral* probability measure \tilde{P} , not the "true" measure P .

In the course of checking that \mathcal{P} satisfies (13) one discovers that

$$v_z(t, h_t^{-1}(y)) = \sigma y P_y(t, y),$$

and hence that $\pi(t) = y(t)P_y(t, y(t))$. Recall that this is the *monetary value* invested in stock, so the corresponding *number of shares* is $\pi(t)/y(t) = P_y(t, y(t))$, which is the strategy $\varphi(t)$ derived earlier.

As a final remark, note that the fundamental solution to the heat equation (22) is

$$v_0(t, x) = \frac{1}{\sqrt{2\pi(T-t)}} e^{-r(T-t)} e^{\frac{-x^2}{2(T-t)}}.$$

The explicit representation (15) for $P(t, y)$ follows from this and (24).

The reader can consult [3] for a detailed exposition of martingale-based approaches to option pricing and other problems in financial economics.

Bibliography

- [1] F. Black and M. Scholes, The pricing of options and corporate liabilities, *J. Political Economy* 81(1973) 637-659
- [2] J.C. Cox and M. Rubinstein, *Options Markets*, Prentice Hall, Englewood Cliffs, 1985

- [3] I. Karatzas, Optimization problems in the theory of continuous trading, *SIAM J. Control and Optimization* 27 (1989), 1221-1259